Knot Floer homology and Pong Algebras

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This is a joint work with Peter Ozsváth.

Overview

1. Knot Floer homology
2. Bordered algebras and knot invariants
3. Pong Algebra

Knot Floer homology

Given a knot $K \subset S^3$ the knot Floer homology $H(K)$ is bigraded over $\mathbf{Z}[u, v]$.

In this lecture we will mostly use mod 2 coefficients: $\mathbf{F}[u, v]$ where $\mathbf{F} = \mathbf{Z}/2\mathbf{Z}$.

There are different versions with increasing order of complexity :

- $u = v = 0$ corresponds to the simplest knot Floer homology $\widehat{HFK}(K)$.
- $v = 0$ is denoted as $HFK^-(K)$. This is related to the algebras $B(2n, n, S)$ Ozsváth-Sz: *Kauffman states, bordered algebras, and a bigraded knot invariant*
- $u \cdot v = 0$ is the version we will concentrate on the first part of the lecture. This is related to some (curved) bordered algebras $B(2n, n)$ and $C(2n, n)$. Ozsváth-Sz: *Bordered knot algebras with matchings* Ozsváth-Sz: *Algebras with matchings and knot Floer homology*
- The general case over $\mathbf{F}[u, v]$ can be viewed as a filtrated version $HFK^-(K)$.

  This is related to the Pong algebra $P(2n, n-1)$ that we will discuss later in the lecture.

Some properties and applications of knot Floer homology:

- $u = v = 0$: A bigraded homology theory $\widehat{HFK}(K) = \oplus_{i,j} H_{i,j}(K)$. The Euler-characteristic is the Alexander polynomial:

$$\sum_{i,j} (-1)^j dim(H_{i,j}(K)) \cdot t^i = \Delta_K(t)$$

  It computes the Seifert genus of K (it is also an unknot-detector):

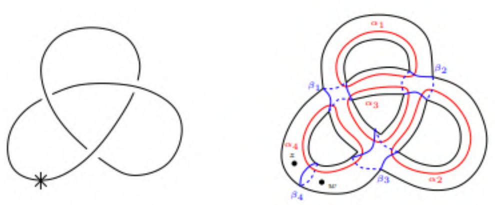
$$max\{i | \oplus_j H_{i,j} \neq 0\} = g(K)$$

  It detects fibered-knots (Ni, see also Ghiggini and Juhász): K is fibered if and only if $dim(\oplus_j H_{g(K),j}) = 1$
- $v = 0$ version can be used to construct an integer-valued knot invariant $\tau(K)$ that bounds the smooth four-ball genus: $g_4(K) \geq |\tau(K)|$
- $u \cdot v = 0$ can be used to compute (the hat version) Heegaard Floer homology of Dehn surgeries $\widehat{HF}(S^3_{p/q}(K))$.
- The full version can be used to compute the plus (and minus) versions Heegaard Floer homology of Dehn surgeries $HF^+(S^3_{p/q}(K))$.

Constructing the knot Floer complex:

- The original construction for $K \subset Y^3$ (where Y^3 is a closed oriented three-manifold) uses two-pointed Heegaard diagrams (Rasmussen and Ozsváth-Sz.)
- For knots in S^3 a combinatorial construction (and a combinatorial way to compute) is due to Manolescu-Ozsváth-Sarkar. (Grid diagrams, Grid homology)
- There is also a version for $K \subset S^3$ that uses knot projections. (this is the version we will study in this lecture).

Figure: A two-pointed Heegaard diagram for the left-handed trefoil

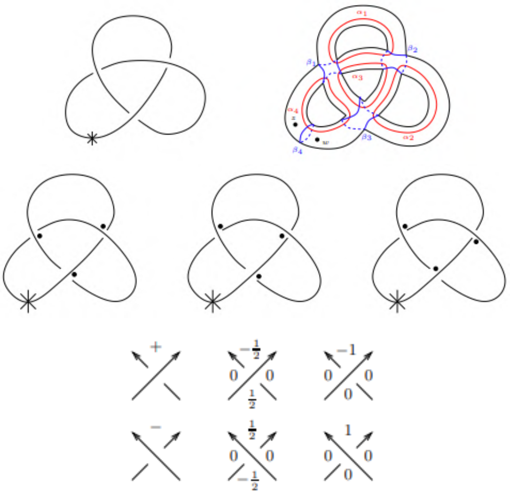


For such a diagram we can associate a complex manifold and two half-dimensional tori:

- $\Sigma_g \to Sym^g(\Sigma_g)$
- $\alpha_1, \cdots, \alpha_g \;\; \to \;\; T_\alpha = \alpha_1 \times \cdots \times \alpha_g$
- $\beta_1, \cdots, \beta_g \;\; \to \;\; T_\beta = \beta_1 \times \cdots \times \beta_g$
- $z \to z \times Sym^{g-1}(\Sigma_g)$
- $w \to w \times Sym^{g-1}(\Sigma_g)$

The generators of the knot Floer complex are given by the intersection-points $T_\alpha \cap T_\beta$.

Figure: Heegaard diagram, Generators, Bigrading



How to compute the boundary map:
Main idea: Degenerations.

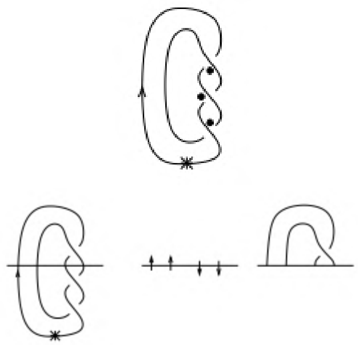
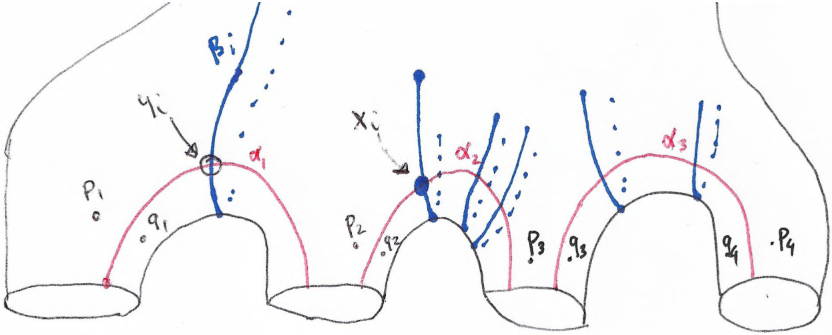


Figure: Upper diagram gives a D-structure over $B(4, 2, \{1, 2\})$

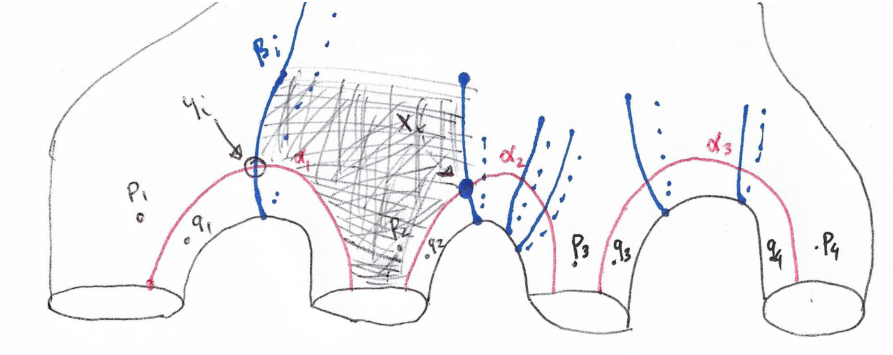
First approach: Use Heegaard diagrams and holomorphic geometry as inspiration to develop an algebraic knot invariant.

Figure: Upper Heegaard diagram near the boundary



The reference points p_i, q_i near the boundary are used to compute the weights of homotopy classes.

Figure: A homotopy class that connects x and y with weight $w = (0, 1/2, 0, 0)$



Setting up the algebra:

- $y = t$ line intersects the knot projection at $2n$ points: $B(2n, n)$.
- There are n regions where the Kauffman states are not yet given: we record this by $0 \leq i_1 < \cdots < i_n \leq 2n$. (idempotents in the algebra).
- In the above example x corresponds to $(1, 3)$ and y to $(2, 3)$.
- There is also a subalgebra $C(2n, n)$ where $1 \leq i_1, \; i_n \leq 2n - 1$
- Homotopy classes have intersection numbers at the degenerating regions. Averaging them we get a weight in $\frac{1}{2}\mathbf{Z}$ for each strand.
- Algebra elements have weights $w(a) \in (\frac{1}{2}\mathbf{Z})^{2n}$ and $w(a \cdot b) = w(a) + w(b)$.
- A special algebra element moves one coordinate $k - 1$ to k, have weight $1/2$ at strand k (and 0 at other strands): R_k. (similarly L_k moves k to $k - 1$.
- U_i have weight 1 at strand k and zero weight at other strands
- There are also relations, for example $R_k R_{k+1} = 0$, $L_{k+1} L_k = 0$.
- Half of the strands are oriented upwards, this is recorded by the set $S \subset \{1, ..., 2n\}$
- For $i \in S$ we have a central algebra element C_i with $d(C_i) = U_i$.

Figure: Examples of algebra elements

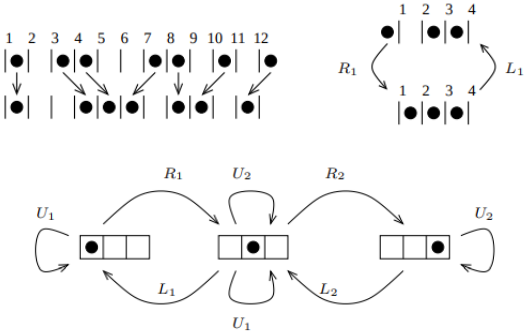
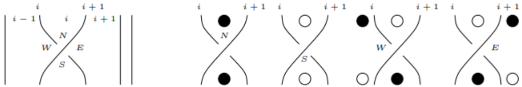
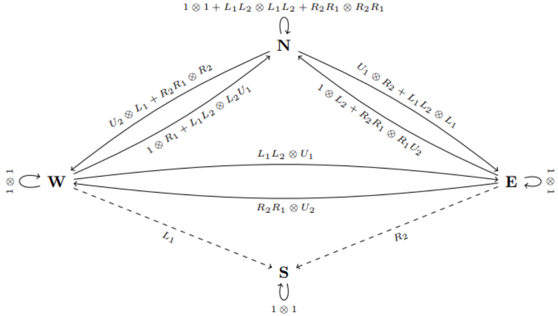


Figure: Some actions for a *DA*-bimodule for a positive crossing





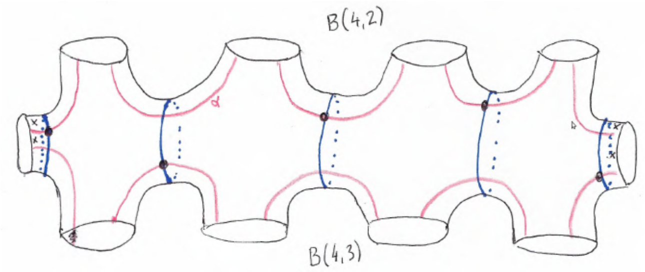
We constructed DA-bimodules for minimums, maximums, where the formulas were inspired by studying certain holomorphic disks in S^2.

Theorem

The corresponding bigraded knot invariant is well defined. Two different knot projection of an oriented knot K gives the same bigraded homology groups over $\mathbf{F}[u]$.

The proof uses (repeatedly) an algebraic result (that was also inspired by Heegaard diagrams):
There is a DD-bimodule over $B(m, k)$ and $B(m, m+1-k)$ that has an AA-bimodule inverse.

Figure: Heegaard diagram for an invertible DD bimodule over $B(m,k)$ and $B(m,m+1-k)$



Role of C_i: dealing with boundary degenerations from the upper diagram.

A different approach: Matchings and curved D-structures.

Figure: Strands 1 and 3 are matched, and strands 2 and 4 are matched

For example if strand 1 is matched to strand 3, and 2 is matched to 4 then

$$D^2 = U_1 U_3 + U_2 U_4$$

This leads to a similar algebraic knot invariant over

$$C(2,1) = \mathbf{F}[u,v]/u \cdot v = 0$$

Variations:

Signs: A signed version of the curved bimodules gives a bigraded knot invariant over $\mathbf{Z}[u, v]$. Question about torsions: Is there a knot $K \subset S^3$ where these homology groups are not free Abelian groups?

Working out the holomorphic degenerations:

Theorem

Knot Floer homology $HKF^-(K)$ over $\mathbf{F}[u, v]/(u \cdot v = 0)$ is isomorphic to the curved algebraic knot invariant.

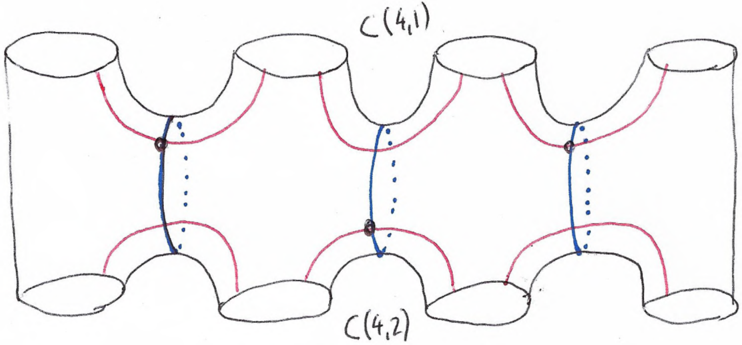
The proof of this result uses the Bordered Heegaard Floer homology package of Lipshitz, Ozsváth and Thurston.

The curved construction can be extended from knots to links:
This gives a method to compute Link Floer homology (and also the Thurston norm).

Main question: How to extend the above results to the "full" Knot Floer homology?

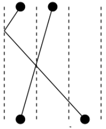
What is the main difficulty:

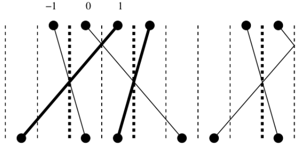
Figure: Heegaard diagram for a DD bimodule over $C(m,k)$, $C(m, m-1-k)$ that is (unfortunately) not invertible



A new algebra to the rescue: Pong algebra $P(m,k)$.

Figure: An algebra element a_1 in $Pong(4,2)$ that corresponds to the strands $1 \to -2$, $2 \to 1$. Left picture: pong diagram. Right picture: an equivariant lift





Weight

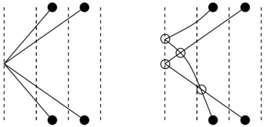
The weight of this algebra element is $(1,1,1/2,0)$. For each wall it is $1/2$ times the number of times the wall is crossed. We can also multiply with polynomials in $\mathbf{F}[u_1,\dots,u_4]$, where for example the weight of $u_1^2 u_4$ is $(2,0,0,1)$.

Derivation

Each term in $d(a)$ corresponds to resolving one of the crossings (or left or right turns) and multiply with a monomial in $\mathbf{F}[u_1,\dots,u_m]$ so that
$Weight(a) = Weight(d(a))$
Example: $d(a_1) = u_1 \cdot b_1 + u_2 \cdot b_2$ where b_1 corresponds to strands $1 \to 3$, $2 \to 1$, and b_2 corresponds to $1 \to 0$, $2 \to 3$.

A more complicated example:

Figure: An algebra element a_2 with crossing number $Cr(a_2) = 4$



In the derivation we also require that for all the terms in $d(a)$ the crossing number is equal to $Cr(a) - 1$. (and delete terms with $Cr(b) < Cr(a) - 1$.)

Example:

$$d(a_2) = u_1 u_2 \cdot b_1 + u_1 u_2 u_3 \cdot b_2 + b_3$$

- $b_1 = (2,3) \to (2,-2)$
- $b_2 = (2,3) \to (-2,3)$
- $b_3 = (2,3) \to (-2,-1)$

The fourth term: $u_1 u_2^2 b_4$ with $b_4 = (2,3) \to (3,2)$ is deleted since $1 = Cr(b_3) < Cr(a_2) - 1 = 3$.

Multiplication in the Pong algebra

We use concatenations of strands and also require that

$$Weight(a \cdot b) = Weight(a) + Weight(b)$$

and

$$Cr(a \cdot b) = Cr(a) + Cr(b)$$

Example: If $a = (1 \to 2)$ and $b = (2 \to 1)$ then

$$a \cdot b = u_2 \cdot (1 \to 1)$$

Example: If a is a left corner $a = (1 \to 0)$ then using the weights it seems we should get $a \cdot a = u_1^2 (1 \to 1)$ however

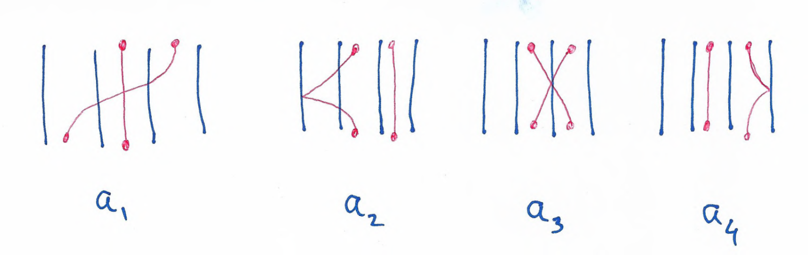
- $Cr(a) = 1$
- $Cr(u_1^2(1 \to 1)) = Cr((1 \to 1)) = 0$

So in this case we have $a \cdot a = 0$.

The homological grading on $P(m,k)$ is defined by the crossing number.
Relationship between $P(m,k)$ and $C(m,m-1-k)$: The special idempotents are given by k element subsets of $\{1,...,m-1\}$.
The complement gives the correspondence between specal idempotents in $P(m,k)$ and $C(m,m-1-k)$.

When computing the homology of $P(m,k)$ it is easy to check that $H_0(P(m,k))$ is isomorphic to $C(m,m-1-k)$.

Figure: Some elements in $P(4,2)$ with homological grading 1.



$$d(a_1) = L_2L_3, \quad d(a_2) = u_1u_2, \quad d(a_3) = u_3, \quad d(a_4) = u_4$$

Theorem

There is a grading preserving isomorphism

$$H_*(P(m,k)) = C(m, m-1-k) \otimes \mathbf{F}[t]$$

where t has homological grading $2k$.

The central element $\Omega \in P(m,k)$ that corresponds to the variable t has $w(\Omega) = (1, ..., 1)$ and $Cr(\Omega) = 2k$.

It can be written as a sum: where $k-1$ strands are unmoving and 1 strand is either

- $i \to i + 2m - 2$ (start rights, meets a right-wall then a left wall and comes back)
- $i \to i - 2m + 2$ (left-wall, right-wall and comes back)

This also means that $P(m,k)$ gives a model for an A^∞ structure on $C(m, m-1-k)$ (with a non-trivial μ_{2k+2} action).

Work in progress: Extend the algebraic knot invariant by replacing $C(2n,n)$ with $P(2n, n-1)$

Note that $P(2,0) = \mathbf{F}[u_1, u_2]$.

Example: $C(4,2)$ and $P(4,1)$.

Figure: The closed algebra element Ω in $P(4,1)$.

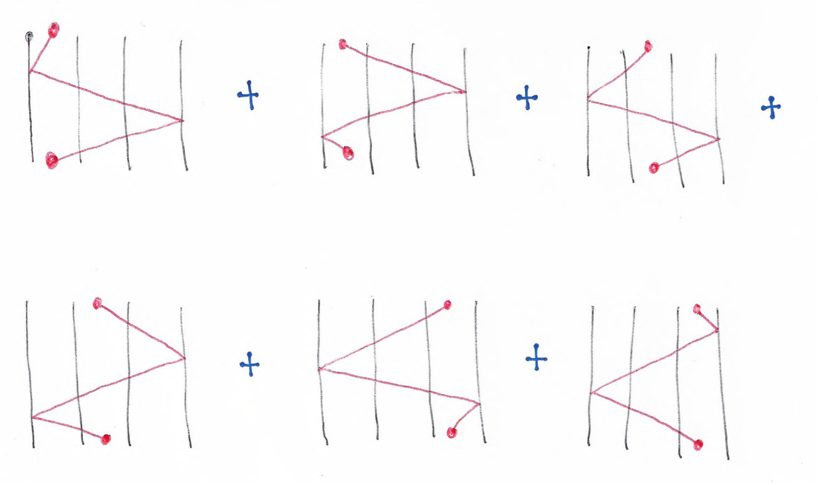


Figure: Heegaard diagram for an invertible DD bimodule over $C(m,k)$ and $P(m,k)/u_i = 0$

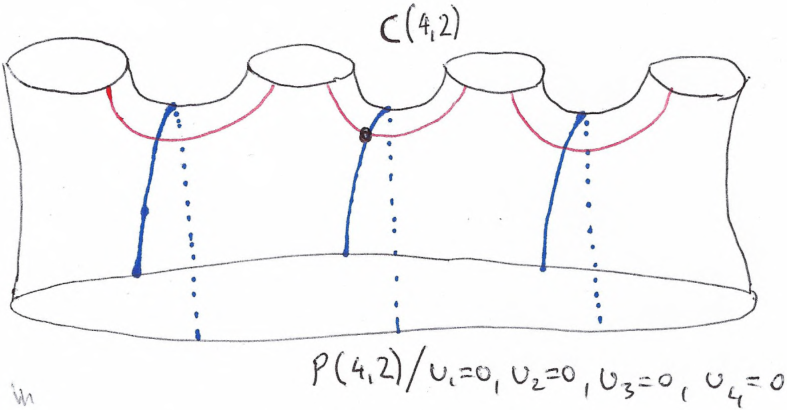
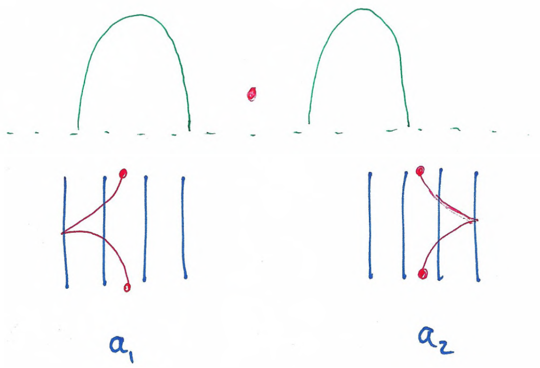
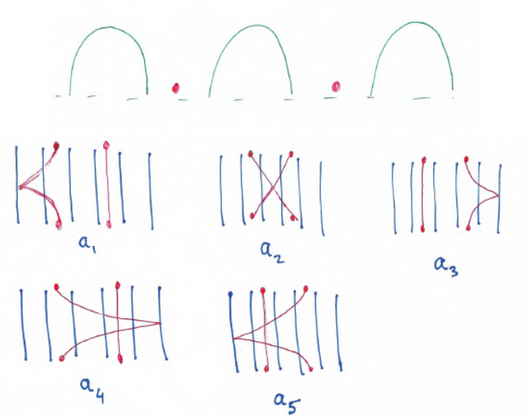


Figure: The upper diagram for 2 maximums



$$D(X) = (a_1 + a_2) \otimes X$$

$$D^2(X) = (u_1 u_2 + u_3 u_4 + \Omega) \otimes X$$



$$D(X) = (a_1 + a_2 + a_3 + a_4 + a_5) \otimes X$$

Exercise: Show that

$$D^2(X) = (u_1 u_2 + u_3 u_4 + u_5 u_6 + \Omega) \otimes X$$