Anchored foams and annular homology

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Let $L \subset \mathbb{R}^3$ be an oriented link with diagram D. Khovanov defines a chain complex $C_{*,*}(D)$ of graded abelian groups.

Theorem (Khovanov)

The chain homotopy class of $C_{*,*}(D)$ is an invariant of L, and its graded Euler characteristic is the Jones polynomial of L.

- $C_{*,*}(D)$ is constructed combinatorially from D.
- A key ingredient is a *2D-TQFT* (equivalently, a Frobenius algebra).
- A Frobenius algebra is a pair (A, R) where R is a commutative ring and A is an R-algebra, with maps

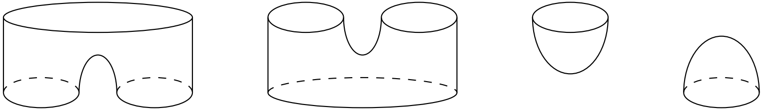
$$m\colon A \otimes A \to A \qquad \Delta\colon A \to A \otimes A$$
$$\eta\colon R \to A \qquad \varepsilon\colon A \to R$$

satisfying certain properties.

Khovanov homology

(A, R) yields a $2D$ TQFT $\mathcal{F} = \mathcal{F}_A$ as follows:

- If C consists of k circles, set $\mathcal{F}(C) = A^{\otimes k}$.
- For a cobordism S from C_0 to C_1, define an R-linear map $\mathcal{F}(S)\colon \mathcal{F}(C_0) \to \mathcal{F}(C_1)$ by writing S as a union of elementary pieces:



  and assigning the maps

$$m\colon A \otimes A \to A, \qquad \Delta\colon A \to A \otimes A, \qquad \eta\colon R \to A, \qquad \varepsilon\colon A \to R$$

  accordingly.

- This assembles into a functor $\mathcal{F}\colon 2\,\mathrm{Cob} \to R-\mathrm{mod}$.

Some relevant Frobenius algebras:

$$\frac{\mathbb{Z}[X]}{(X^2)} = H^*(\mathbb{CP}^1; \mathbb{Z})$$

$$\frac{\mathbb{Z}[E_1, E_2, X]}{(X^2 - E_1 X + E_2)},$$

the $U(2)$-equivariant cohomology of $\mathbb{CP}^1$. This yields *equivariant* or *universal* Khovanov homology.

Equivariant versions of link homology have been developed:

- Mackaay-Vaz in the $sl(3)$ setting [MV07].
- Krasner for $sl(n)$ Khovanov-Rozansky homology [Kra10].
- Wu for colored $sl(n)$ homology [Wu12].

Recent constructions by Ehrig-Tubbenhauer-Wedrich [ETW18] of $sl(n)$ homology via Robert-Wagner [RW20] closed foam evaluation are naturally equivariant.

Annular link homology

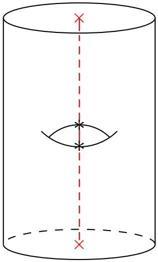
- Asaeda-Przytycki-Sikora [APS04] defined a homology theory for links in interval bundles over surfaces.
- The special case of the thickened annulus is known as *annular Khovanov homology* or *annular APS homology*.
- Let $\mathbb{A} := S^1 \times [0,1]$ denote the annulus.
- For a link $L \subset \mathbb{A} \times [0,1]$, project onto $\mathbb{A} \times \{0\}$ to obtain a diagram.
- Form the cube of resolutions as usual, with all smoothings drawn in $\mathbb{A}$.
- Apply the TQFT $\bigcirc \mapsto \mathbb{Z}[X]/(X^2)$.
- Winding number induces a filtration which is respected by the differential. The annular chain complex is the associated graded.
- Annular homology is triply graded.
- An equivariant version of annular homology was defined in earlier work using a filtration as above.
- We define equivariant $sl(2)$ and $sl(3)$ annular homology via closed foam evaluation, in the spirit of Robert-Wagner [RW20].

A., *Equivariant annular Khovanov homology.* arXiv:2008.00577

Annular link homology

Identify the interior of $\mathbb{A}$ with the punctured plane $\mathcal{P} := \mathbb{R}^2 \setminus \{(0,0)\}$.
Links in $\mathbb{A} \times [0,1]$ correspond to links in $\mathcal{P} \times [0,1]$.
Let $L = \{(0,0)\} \times \mathbb{R} \subset \mathbb{R}^3$ denote the z-axis (*anchor line*).
We will define a suitable TQFT via universal construction:

- A module $\langle C \rangle$ for a collection of simple closed curves $C \subset \mathcal{P}$.
- A map $\langle S \rangle : \langle C_0 \rangle \to \langle C_1 \rangle$ for a (generic) cobordism $S \subset \mathbb{R}^2 \times [0,1]$ from C_0 to C_1.



Universal construction

Idea (Blanchet-Habegger-Masbaum-Vogel): invariants of closed n-dimensional objects can yield TQFT for $(n-1)$-dimensional objects.

This was used by Robert-Wagner, who give an evaluation of closed foams which categorifies Murakami-Ohtsuki-Yamada (MOY) calculus.

In our $sl(2)$ annular setting:

Definition

An *anchored surface* is a closed surface $S \subset \mathbb{R}^3$ which is transverse to the line L. Intersection points $S \cap L$ (called *anchor points*) come with a *labeling*

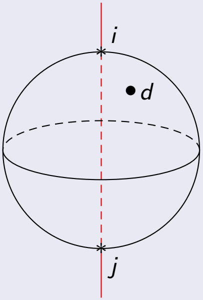
$$\ell \colon S \cap L \to \{1, 2\}.$$

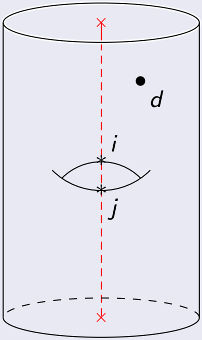
Components of S may be decorated by finitely many dots.

We also consider *anchored cobordisms* $S \subset \mathbb{R}^2 \times [0, 1]$, with $\partial S \subset \mathcal{P} \times \{0, 1\}$ and points in $S \cap L$ carrying labels in $\{1, 2\}$.

Universal construction

Examples





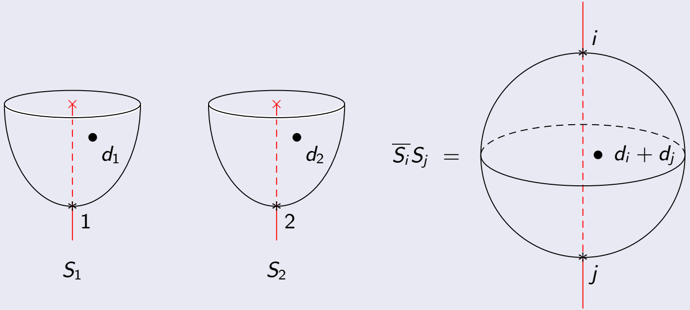
$i, j \in \{1, 2\}$

Suppose we have an evaluation $\langle S \rangle \in R$ for closed anchored surfaces, valued in some commutative ring R.

- Let $C \subset \mathcal{P}$ be a collection of simple closed curves.
- Let $\mathrm{Fr}(C)$ be the free R-module with basis all anchored cobordisms $S \subset \mathbb{R}^2 \times (-\infty, 0]$ with $\partial S = C$.
- Define
$$(-,-)\colon \mathrm{Fr}(C) \times \mathrm{Fr}(C) \to R$$
by
$$(S_1, S_2) = \langle \overline{S_1} S_2 \rangle$$
where $\overline{S_1}$ is the reflection of S_1 through $\mathbb{R}^2 \times \{0\}$.
- Set
$$\ker((-,-)) = \{x \in \mathrm{Fr}(C) \mid (x, y) = 0 \text{ for all } y \in \mathrm{Fr}(C)\},$$
and define the *state space*
$$\langle C \rangle = \mathrm{Fr}(C) / \ker((-,-)).$$

Universal construction

Example

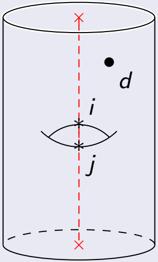


For an anchored cobordism $S\colon C_0 \to C_1$, we immediately obtain a map

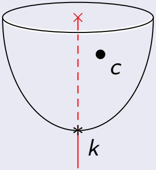
$$\langle S \rangle : \langle C_0 \rangle \to \langle C_1 \rangle$$

defined by $\langle S \rangle ([S']) = [SS']$. This assignment is functorial.

Example

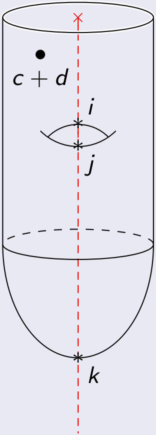


S



S'

$\langle S \rangle ([S']) \ =$



Evaluation of anchored surfaces

Let S be an anchored surface.

- Let $\mathrm{Comp}(S)$ denote the components of S.
- A *coloring* of S is a function

$$c\colon \mathrm{Comp}(S) \to \{1, 2\}.$$

- Let $\mathrm{adm}(S)$ denote the set of colorings.

Consider the ring

$$R_\alpha := \mathbb{Z}[\alpha_1, \alpha_2].$$

For $c \in \mathrm{adm}(S)$, will define the *evaluation*

$$\langle S, c\rangle \in R_\alpha[(\alpha_1 - \alpha_2)^{-1}],$$

and then set

$$\langle S\rangle := \sum_{c\in\mathrm{adm}(S)} \langle S, c\rangle$$

Evaluation of anchored surfaces

Fix a closed anchored surface S and a coloring c.

- For $i = 1, 2$, let $d_i(c)$ denote the number of dots on components colored i.
- Let $S_2(c)$ denote the union of the 2-colored components.
- For $p \in S \cap L$, let $\ell(p)$ denote the label of p (independent of c).
- Let $c(p)$ denote the color of the component containing p (depends on c).

Define

$$\langle S, c \rangle = (-1)^{\chi(S_2(c))/2} \frac{\alpha_1^{d_1(c)} \alpha_2^{d_2(c)}}{(\alpha_1 - \alpha_2)^{\chi(S)/2}} \left(\prod_p (\alpha_{c(p)} - \alpha_{\ell(p)}) \right)^{1/2} .$$

Evaluation of anchored surfaces

$$\langle S, c\rangle = (-1)^{\chi(S_2(c))/2} \frac{\alpha_1^{d_1(c)} \alpha_2^{d_2(c)}}{(\alpha_1 - \alpha_2)^{\chi(S)/2}} \left(\prod_p (\alpha_{c(p)} - \alpha_{\ell(p)}) \right)^{1/2} .$$

The square root is defined as follows.

- If a component $S' \subset S$ is colored $i \in \{1, 2\}$ and has an anchor point labeled i, then $\langle S, c\rangle = 0$.
- Otherwise, S' has $2k \geq 0$ anchor points with label $j \neq i$, and it contributes $(\alpha_i - \alpha_j)^k$.

Set

$$\langle S\rangle = \sum_{c \in \mathrm{adm}(S)} \langle S, c\rangle .$$

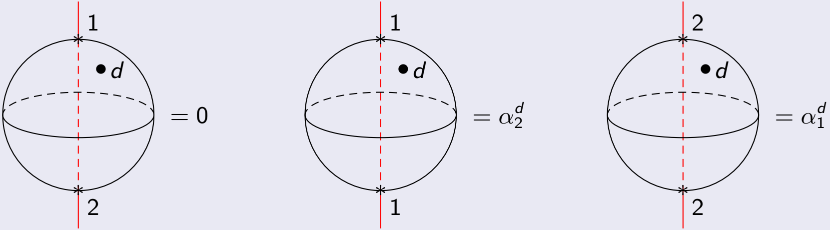
If $S = S_1 \sqcup \cdots \sqcup S_n$, then $\langle S\rangle = \langle S_1\rangle \cdots \langle S_n\rangle$.
To see that $\langle S\rangle \in \mathbb{Z}[\alpha_1, \alpha_2]$:

- Only a 2-sphere $\mathbb{S}^2$ has positive Euler characteristic.
- If such a component is disjoint from L then the sum over its two colorings is $\frac{\alpha_1^d - \alpha_2^d}{\alpha_1 - \alpha_2} \in \mathbb{Z}[\alpha_1, \alpha_2]$.
- If such a component intersects L then the contributions from anchor points cancel with the denominator.

Evaluation of anchored surfaces

$$\langle S, c \rangle = (-1)^{\chi(S_2(c))/2} \frac{\alpha_1^{d_1(c)} \alpha_2^{d_2(c)}}{(\alpha_1 - \alpha_2)^{\chi(S)/2}} \left(\prod_p (\alpha_{c(p)} - \alpha_{\ell(p)}) \right)^{1/2}$$

Example



Note that $\langle S \rangle$ is not a symmetric polynomial.

Evaluation of anchored surfaces

Remark

If $S \cap L = \varnothing$, then $\langle S \rangle$ is the usual evaluation of a closed surface in equivariant (universal) $sl(2)$ link homology.

In this case, $\langle S \rangle$ is a symmetric polynomial,

$$\langle S \rangle \in \mathbb{Z}[E_1, E_2] \subset \mathbb{Z}[\alpha_1, \alpha_2],$$

with

$$E_1 = \alpha_1 + \alpha_2, \quad E_2 = \alpha_1 \alpha_2.$$

The Frobenius algebra assigned to a contractible circle is

$$\frac{\mathbb{Z}[\alpha_1, \alpha_2, X]}{(X^2 - E_1 X + E_2)},$$

and $X^2 - E_1 X + E_2 = (X - \alpha_1)(X - \alpha_2)$. It can instead be defined over the subring $\mathbb{Z}[E_1, E_2]$.

Gradings

State spaces are bigraded. For an anchored cobordism S,

$$\mathrm{qdeg}(S) = -\chi(S) + 2 \cdot \#\text{dots} + \#\text{anchor points}.$$

We have $\mathrm{qdeg}(S) = \deg(\langle S \rangle)$, where $\deg(\alpha_1) = \deg(\alpha_2) = 2$.

There is a second grading adeg coming from intersections with L.

- Label the anchor points $1, \ldots, m$ from bottom to top.
- Set $\mathrm{adeg}(S) = \sum_{i=1}^{m} (-1)^{i+\ell(i)}$

	label 1	label 2
i odd	1	-1
i even	-1	1

Set R_α to be concentrated in annular degree zero.

Lemma

If S is a closed anchored surface, then $\langle S \rangle = 0$ or $\mathrm{adeg}(S) = 0$.

Extend adeg to anchored cobordisms (with boundary) and to state spaces $\langle C \rangle$.

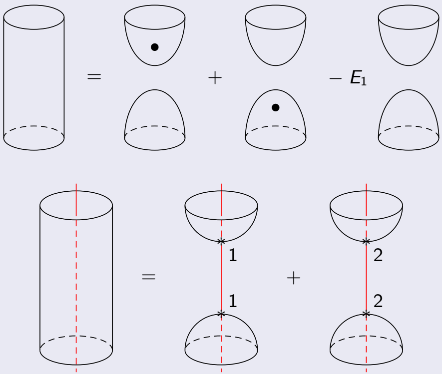
Lemma

Let $S\colon C_0 \to C_1$ be an anchored cobordism. The map $\langle S \rangle : \langle C_0 \rangle \to \langle C_1 \rangle$ is bi-homogeneous of degree $(\mathrm{qdeg}(S), \mathrm{adeg}(S))$.

State spaces

We have neck-cutting relations for anchored surfaces:

Lemma



which allow us to identify state spaces as follows.

State spaces

Theorem

Let $C \subset \mathcal{P}$ consist of n contractible circles and m non-contractible circles. Then the state space $\langle C \rangle$ is a free R_α-module of graded rank

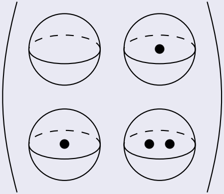
$$(q+q^{-1})^n(a+a^{-1})^m.$$

Proof.

By neck-cutting, $\langle C \rangle$ is spanned by disk cobordisms where each

- disk with contractible boundary is disjoint from L and carries 0 or 1 dots ,
- disk with non-contractible boundary intersects L once, with label 1 or 2.

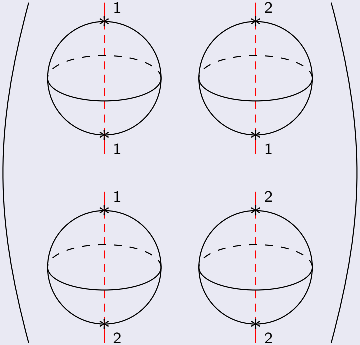
This forms a basis by computing the bilinear form.



$$= \begin{pmatrix} 0 & 1 \\ 1 & E_1 \end{pmatrix}$$

State spaces

Proof continued



$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Annular link homology

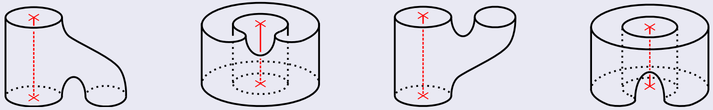
- We have a functor $\langle - \rangle : \mathrm{ACob} \to R_\alpha - \mathrm{ggmod}$ from the category of anchored cobordisms into the category of bigraded R_α-modules.
- Applying $\langle - \rangle$ to the cube of resolutions yields annular link homology.
- We can restrict to the sub-category ACob' consisting of cobordisms disjoint from L (all cobordisms in the cube of resolutions are of this form).
- Another functor $\mathcal{G}_\alpha : \mathrm{ACob}' \to R_\alpha - \mathrm{ggmod}$ was constructed earlier.

Theorem

The functors $\langle - \rangle : \mathrm{ACob}' \to R_\alpha - \mathrm{ggmod}$ *and* $\mathcal{G}_\alpha : \mathrm{ACob}' \to R_\alpha - \mathrm{ggmod}$ *are naturally isomorphic.*

Proof.

It suffices to check the four elementary cobordisms:



□

A., *Equivariant annular Khovanov homology.* arXiv:2008.00577

sl(3) foams

There are two types of $sl(3)$ foams:

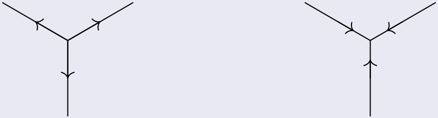
- oriented foams, first appearing in Khovanov's categorification of the $sl(3)$ link polynomial [Kho04].
- unoriented foams, studied by Khovanov-Robert [KR21], related to graph colorings and gauge-theoretic constructions due to Kronheimer-Mrowka [KM19].

We consider both in the annular setting but will focus on oriented foams.

In $sl(3)$ homology, circles in the plane are replaced by *webs*:

Definition

An oriented $sl(3)$ web is an oriented trivalent graph $\Gamma \subset \mathbb{R}^2$ (which may contain closed loops) such that each vertex is a source or a sink.

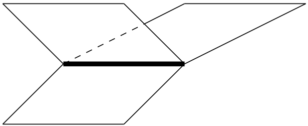


Cobordisms between circles are replaced by *foams* ("cobordisms" between webs).

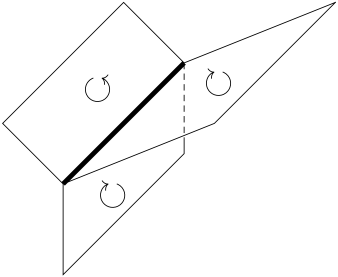
One needs modules $\langle \Gamma \rangle$ and functorial maps induced by foams.

Oriented sl(3) foams

An *oriented sl*(3) foam is a 2-dimensional CW complex with singularities of the form $Y \times [0, 1]$.



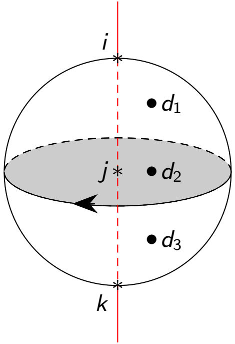
Two-dimensional cells (facets) must be oriented as follows



Oriented sl(3) foams

We will consider *anchored foams*:

- A foam $F \subset \mathbb{R}^3$ which intersects the line L transversely in the interior of its facets.
- Anchor points $F \cap L$ carry fixed labels in $\{1, 2, 3\}$.
- Facets may carry dots.



$i, j, k \in \{1, 2, 3\}$

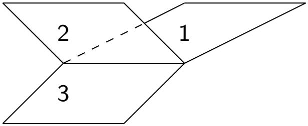
Foam evaluation

Via universal construction, an evaluation $\langle F \rangle$ of closed anchored foams yields state spaces $\langle \Gamma \rangle$ for webs $\Gamma \subset \mathcal{P}$ and functorial maps induced by foams with boundary.

An *admissible coloring* of F is a function

$$c\colon \{\text{facets of } F\} \to \{1, 2, 3\}$$

such that all three colors appear near singularities:



- Let $\mathrm{adm}(F)$ denote the set of admissible colorings.
- For $c \in \mathrm{adm}(F)$ and $1 \leq i \neq j \leq 3$, let $F_{ij}(c)$ denote the union of i and j colored facets.
- $F_{ij}(c)$ is a closed, orientable surface in $\mathbb{R}^3$.
- For $i \in \{1, 2, 3\}$, let $d_i(c)$ denote the number of dots on i-colored facets.

Foam evaluation

For $i \in \{1,2,3\}$, let $i', i'' \in \{1,2,3\}$ denote the complementary elements, so that $\{i, i', i''\} = \{1,2,3\}$.
We define an evaluation $\langle F \rangle \in \mathbb{Z}[x_1, x_2, x_3]$:

$$P(F,c) = \prod_{i=1}^{3} x_i^{d_i(c)}$$

$$Q(F,c) = \prod_{1 \le i < j \le 3} (x_i - x_j)^{\chi(F_{ij}(c))/2}$$

$$\widetilde{Q}(F,c) = \left(\prod_p (-1)^{c(p)-1} (x_{c(p)} - x_{\ell(p)'})(x_{c(p)} - x_{\ell(p)''}) \right)^{1/2}$$

$$\langle F, c \rangle = (-1)^{s(F,c)} \frac{P(F,c)\widetilde{Q}(F,c)}{Q(F,c)}$$

$$\langle F \rangle = \sum_{c \in \mathrm{adm}(F)} \langle F, c \rangle$$

Foam evaluation

$$\widetilde{Q}(F,c) = \left(\prod_p (-1)^{c(p)-1} (x_{c(p)} - x_{\ell(p)'})(x_{c(p)} - x_{\ell(p)''}) \right)^{1/2}$$

- If $c(p) \neq \ell(p)$ for some anchor point p then $\langle F, c \rangle = 0$.
- We may assume every anchor point is colored according to its label.
- Then p contributes

$$\begin{aligned} (x_1 - x_2)(x_1 - x_3) &\text{ if } c(p) = \ell(p) = 1, \\ (x_1 - x_2)(x_2 - x_3) &\text{ if } c(p) = \ell(p) = 2, \\ (x_1 - x_3)(x_2 - x_3) &\text{ if } c(p) = \ell(p) = 3. \end{aligned}$$

- If $\mathrm{an}(i)$ denotes the number of anchor points labeled i, then

$$\mathrm{an}(i) + \mathrm{an}(j) = |F_{ij}(c) \cap L| \text{ is even, and}$$

$$\widetilde{Q}(F,c) = \prod_{1 \leq i < j \leq 3} (x_i - x_j)^{(\mathrm{an}(i)+\mathrm{an}(j))/2}.$$

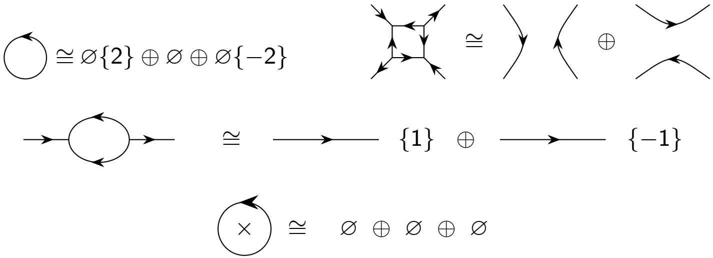
Theorem

For a closed anchored foam F, we have $\langle F\rangle \in \mathbb{Z}[x_1, x_2, x_3]$. If F is disjoint from L, then the evaluation agrees with that of Mackaay-Vaz.

We can form state spaces $\langle\Gamma\rangle$ for webs $\Gamma \subset \mathcal{P}$ as explained earlier. State spaces carry a quantum grading: for a foam cobordism $V\colon \varnothing \to \Gamma$,

$$\mathrm{qdeg}(V) = 2\,(\#\text{dots} + \#\text{anchor points} - \chi(V)) + \chi(\Gamma).$$

We have local web isomorphisms for state spaces:



Using these local isomorphisms we obtain:

Theorem

For any web $\Gamma \subset \mathcal{P}$, the state space $\langle \Gamma \rangle$ is a free graded $\mathbb{Z}[x_1, x_2, x_3]$-module of rank equal to the number of Tait colorings of Γ. Moreover, if Γ is contractible, then the graded rank of $\langle \Gamma \rangle$ equals the Kuperberg polynomial of Γ.

State spaces carry an *annular* grading, valued in

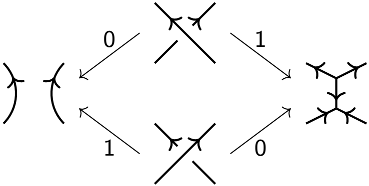
$$\Lambda := \frac{\mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \oplus \mathbb{Z}w_3}{(w_1 + w_2 + w_3)}.$$

For a foam cobordism V, define

$$\mathrm{adeg}(V) = \sum_p s(p) w_{\ell(p)} \in \Lambda.$$

where $s(p) \in \{\pm 1\}$ is the oriented intersection number.

Given an oriented link $L \subset \mathbb{A} \times [0,1]$, form the $sl(3)$ chain complex in the standard way.



Applying $\langle - \rangle$ yields *equivariant annular* $sl(3)$ *homology*. It carries homological, quantum, and annular (Λ) gradings.

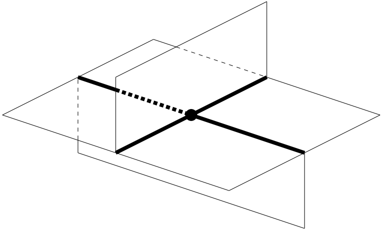
Remark

Queffelec-Rose [QR18] defined (non-equivariant) annular Khovanov-Rozansky $sl(n)$ homology, and show it carries an action of $sl(n)$. The Λ grading is expected from this point of view, but we do not have an $sl(3)$ action in the equivariant setting.

Unoriented sl(3) foams

Unoriented *sl*(3) foams were studied by Khovanov-Robert [KR21]. They are a combinatorial counterpart to gauge-theoretic constructions introduced by Kronheimer-Mrowka [KM19].

Unoriented *sl*(3) foams are cobordisms between trivalent planar graphs. They may have singularities of the form



We also extend Khovanov-Robert foam evaluation to the anchored setting.

Thank You!

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