Instanton L-spaces & splicing

w/ Steven Sivek

Conj: $Y^3 \not\cong S^3 \Rightarrow \exists$ nontrivial $\rho : \pi_1(Y) \longrightarrow SU(2)$

$$\begin{tikzcd} \pi_1(Y) \arrow[r] \arrow[dr, red] & SU(2) \\ & \color{red}{H_1(Y)} \arrow[u, red] \end{tikzcd}$$

Rmk: Suffices to consider prime $\mathbb{Z}HS^3$

Proved for $\begin{cases} \text{Seifert mflds (Fintushel-Stern)} \\ \text{Toroidal mflds (Lidman-Pinzón-Caicedo-Zentner '21)} \end{cases}$

$\Downarrow$ $\updownarrow$ I_*, holonomy perturbations, pillowcase

Thm (Zentner '18): $Y \not\cong S^3 \Rightarrow \exists$ nontrivial $\rho : \pi_1(Y) \to SL(2,\mathbb{C})$

Goal: different proof of LPCZ $\Rightarrow$ Zentner.

Framed instanton homology

$Y^3 \rightsquigarrow I^\#(Y) := \text{"}\tfrac{1}{2}\text{"}$ of $I_*(Y \# T^3)_{S'}$

Def: a $\mathbb{Q}HS^3$ Y is an <u>L-space</u> if $\dim I^\#(Y) = |H_1(Y;\mathbb{Z})|$.

Fact: $\mathbb{Z}HS^3$ Y <u>not</u> an L-space $\Rightarrow \exists$ nontrivial $\rho : \pi_1(Y) \to SU(2)$

$\Leftrightarrow \dim I^\#(Y) > 1$

Def: a <u>splice</u> of knots $\alpha \subseteq A$ and $\beta \subseteq B$ is a 3-mfld

$$Y = A \setminus N(\alpha) \cup_\varphi B \setminus N(\beta)$$

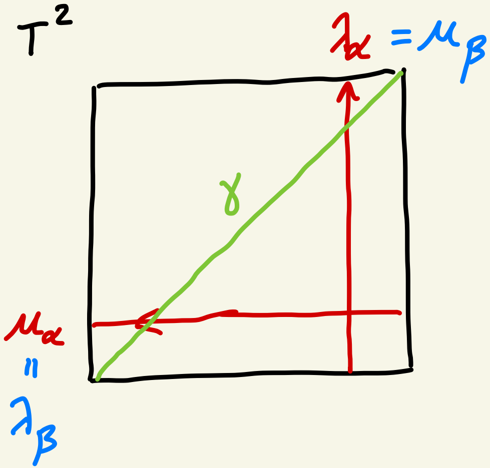
Thm (B-Sivek): Let C be a splice of nontrivial knots $\alpha \subseteq A$ and $\beta \subseteq B$ in homology sphere L-spaces, where φ identifies meridians w/ Seifert longitudes. Then $\dim I^{\#}(C) \geq 5$.
In particular, the $\mathbb{Z}HS^3$ C is not an L-space.

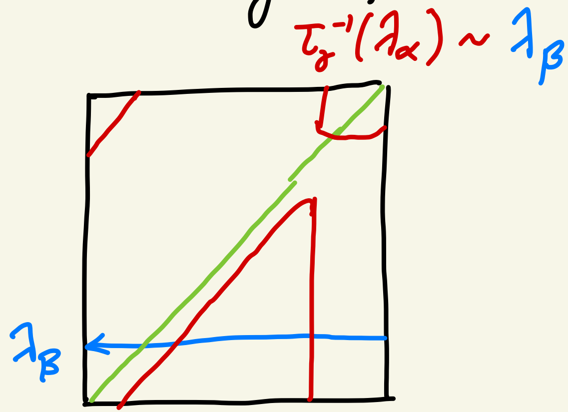
Cor: Y^3 toroidal $\Rightarrow$ $\exists$ nontrivial $\rho: \pi_1(Y) \to SU(2)$

Pf: Assume Y a $\mathbb{Z}HS^3$ $\Rightarrow$ Y is splice of nontrivial $\alpha \subseteq A$, $\beta \subseteq B$ in homology spheres, where meridians are identified w/ longitudes.
$\exists$ degree-1 maps $Y \to A$, $Y \to B$. So, can assume neither $\pi_1(A)$ nor $\pi_1(B)$ admits nontrivial $SU(2)$-rep. $\Rightarrow$ A, B are L-spaces.
$\overset{\text{Thm}}{\Longrightarrow}$ Y not an L-space $\Rightarrow$ $\exists$ nontrivial $\rho: \pi_1(Y) \to SU(2)$ $\square$

Let $C = A \setminus N(\alpha) \cup_{T^2} B \setminus N(\beta)$, $\alpha = \partial \Sigma_\alpha$, $\beta = \partial \Sigma_\beta$

Let $\gamma \subseteq T^2 \subseteq C$ be a curve of slope -1





$C_0(\gamma) \cong A_{-1} \# B_{-1}$, where $A_i = A_i(\alpha)$, $B_i = B_i(\beta)$

$C_1(\gamma) \cong$ a splice S where φ identitifies the Seifert longitudes.

closed $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \subseteq S$, $g(\Sigma) \geq 2$

$\Sigma \rightsquigarrow \mu(\Sigma) : I^{\#}(S)^{\circlearrowleft}$, eigenvalues in $\{2-2g, 4-2g, \ldots, 2g-2\}$

Def: $I^{\#}(S)_{\Sigma}$:= direct sum of (generalized) $2g-2$ and $2-2g$ eigenspaces of S

distinct since $g \geq 2$.

Thm (B-Sivek):

$$\dim I^{\#}(S)_{\Sigma} = 4 \cdot \underline{\dim KHI}(A, \alpha, g(\alpha)) \cdot \underline{\dim KHI}(B, \beta, g(\beta)) \geq 4.$$

$$
\begin{tikzcd}
I^{\#}(C) \arrow[r] \arrow[d] & I^{\#}(A_{-1}\# B_{-1}) \arrow[r, "a"] \arrow[d, hook, "b"] & I^{\#}(S) \arrow[d, "c"] \\
I^{\#}(A_0 \# B) \arrow[r] & I^{\#}(A_0 \# B_{-1}) \arrow[r, hook, "d"] & I^{\#}(A_0 \# B_0)
\end{tikzcd}
$$

0-surgery on pushoff of $\alpha \subseteq \Sigma_\alpha$

<u>Claim:</u> $\operatorname{im} a \cap I^{\#}(S)_\Sigma = 0$

Bottom row is $I^{\#}(A_0) \otimes \big(\underset{\substack{\cong \\ \mathbb{Q}}}{I^{\#}(B)} \to I^{\#}(B_{-1}) \to I^{\#}(B_0)\big)$

b, d are either inj or surj.

WLOG, can assume b is injective. Suppose d is injective.

Say $a(\underset{\neq 0}{x}) \in I^{\#}(S)_\Sigma$. Then $c(a(x)) = d(b(x)) \neq 0$

But $I^{\#}(S)_\Sigma \subseteq \ker c$, since α compresses Σ, contradiction.

Q: What about toroidal Y w/ $|H_1(Y)| > 1$, but small?

Thm (Hanselman-Rasmussen-Watson): Y toroidal and $|H_1(Y)| < 5$
$\Rightarrow$ Y not a Heegaard Floer L-space.

If true for $I^{\#}$, could prove

Conj: $|H_1(Y)| < 5 \Rightarrow \exists$ irreducible $\rho : \pi_1(Y) \to SL(2, \mathbb{C})$
for all but finitely many (known) SFS Y.

$*$

If Y not an L-space $\Rightarrow \exists$ irreducible $\rho : \pi_1(Y) \to SU(2)$