TOROIDAL 3-MANIFOLDS & THE PROPERTIES IN THE L-SPACE CONJECTURE

joint with Steve Boyer & Ying Hu

L-Space Conjecture (Boyer-G-Watson, Juhász)

M a prime, closed, connected, orientable 3-mfld.

Then TFAE:

(1) M is not a Heegaard Floer L-space (M is NLS)

(2) $\pi_1(M)$ is left-orderable (M is LO)

(3) M has a co-orientable taut foliation (M is CTF)

NLS: either $H_1(M)$ is infinite

or M is a $\mathbb{Q}HS$ and

$$\dim_{\mathbb{F}_2} \widehat{HF}(M) > |H_1(M)|$$

LO: $\exists$ total order $<$ on $\pi_1(M)$ s.t.

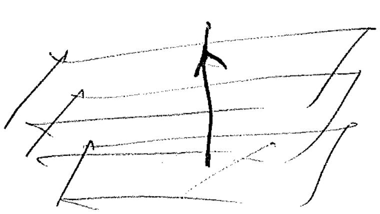
$g < h \Rightarrow fg < fh \quad (\forall f, g, h \in \pi_1(M))$

$\equiv$ $\exists$ faithful rep. $\pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$

$\equiv$ " non-trivial " " "

(Boyer–Rolfsen–Wiest)

CTF: $\exists$ codim 1 foliation $\mathcal{F}$ on M



taut: $\exists$ closed loop in M meeting every leaf of $\mathcal{F}$, transversely.

<u>Remarks</u>. (1) $H_1(M)$ infinite $\Rightarrow$ M is NLS, LO & CTF

(2) CTF $\Rightarrow$ NLS (Ozsváth–Szabó).

(3) Geom. Conj. $\Rightarrow$ M is Seifert fibered, toroidal, or hyperbolic.

We'll focus on case: <u>M toroidal</u>.

$\mathbb{Z}$-Homology Spheres.

<u>Conj.</u> (OS). M a $\mathbb{Z}HS$. Then M is an L-space $\Leftrightarrow$ $M \cong S^3$ or $\Sigma(2,3,5)$.

<u>Thm A</u>. (Eftekhary; Hanselman-Rasmussen-Watson).

<u>A toroidal $\mathbb{Z}HS$ M is NLS.</u>

<u>Thm 1</u> (Boyer-G-Hu). M a toroidal $\mathbb{Z}HS$.

Then (1) <u>M is LO</u>

(2) If $M = X_1 \cup_T X_2$, T incomp. torus, <u>X_2 fibred</u>, then M is CTF.

Remarks. (1) Thm. A & Thm 1(1) hold for M with $|H_1(M)| \leq 4$; & Thm 1(2) if X_1 & X_2 are fibered.

All false for $|H_1(M)| = 5$.

(2) Thm 1(1) $\Rightarrow \exists$ faithful rep. $\pi_1(M) \to \mathrm{Homeo}_+(\mathbb{R})$.

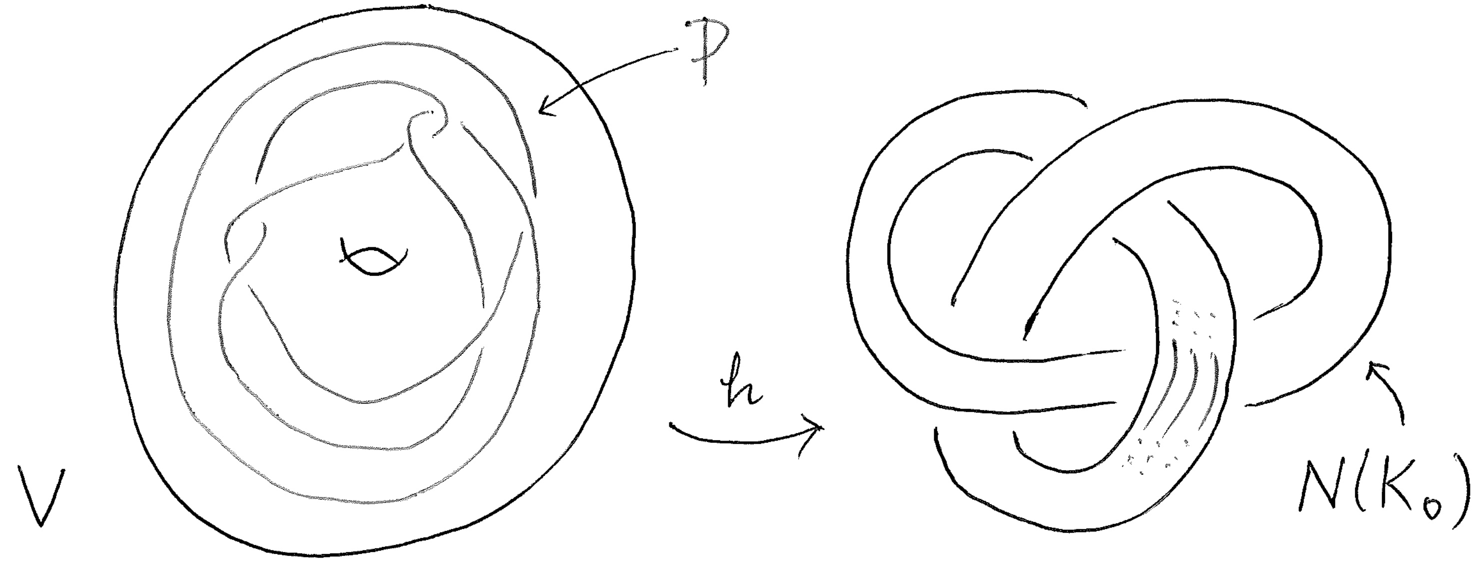
Thm. (Lidman - Pinzon-Caicedo - Zentner). M a toroidal $\mathbb{Z}HS$ $\Rightarrow \exists$ non-trivial rep. $\pi_1(M) \to SU(2)$.

$\exists$ toroidal $\mathbb{Z}HS$s M with no non-trivial rep. $\pi_1(M) \to PSL(2,\mathbb{R})$. (based on Gao).

Satellite knots

K non-trivial knot $\subset S^3$	$X_K = S^3 \setminus \mathring{N}(K)$
torus knot $T(p,q)$	Seifert
satellite knot	toroidal
hyperbolic	hyperbolic

Satellite knot:



$|P \cap \text{meridian disk of } V| \geq 2$ $\qquad$ $h : V \xrightarrow{\cong} N(K_0)$

$K = h(P) = P(K_0) =$ <u>satellite</u> of K_0

$K_0 =$ <u>companion</u> of K

$K \longmapsto \Sigma_n(K)$ = n-fold cyclic branched cover of K.

Conj. (G-Lidman). K a prime satellite knot. Then $\forall n \geq 2$ $\Sigma_n(K)$ is LO & CTF (& ∴ NLS).

Thm 2 (BGH). " " " . Then $\forall n \geq 2$ $\Sigma_n(K)$ is NLS, LO, & CTF if its companion is fibered.

GL-Conj. should hold for links.

Thm 3 (BGH). L prime link, X_L toroidal. Then $\Sigma_2(L)$ is NLS.

Thm (Menasco). A prime non-split alternating link is either $T(2,q)$ or hyperbolic.

OS defined quasi-alternating (QA) link.

L non-split alt. $\Rightarrow$ L QA $\Rightarrow$ L $\mathbb{Z}_2$-Khovanov thin $\xRightarrow[\text{OS}]{}$ $\Sigma_2(L)$ L-space

Manolescu–O.

Thm 4 (BGH). A prime $\mathbb{Z}_2$-Kh. thin link (e.g. a QA link) is either $T(2,q)$ or hyperbolic.

Proof. L $\mathbb{Z}_2$-Kh. thin; not hyperbolic.

OS Thm
Thm 3 $\Big\} \Rightarrow X_L$ Seifert

$\Downarrow$ (Burde-Murasugi)

L braid positive

L strongly quasipositive
$\Sigma_2(L)$ L-space $\Big\} \Rightarrow$ L definite

(Boileau-Boyer-G)

$\Longrightarrow$ L is an ADE +ve Hopf band plumbing
(Baader)

i.e. $T(2,q)$, $q \geq 2$

$T(3,4)$, $T(3,5)$

$P(-2,2,m)$, $m \geq 2$
$P(-2,3,4)$

not Kh thin
(Manion)

not Kh thin
(Khovanov)

X a $\mathbb{Q}H(S^1 \times D^2)$ $(\neq S^1 \times D^2)$

$\exists$ notion of a slope on ∂X $(\cong T^2)$ being $*$-detected, $* \in \{NLS, LO, CTF\}$ for which, if $\mathcal{D}_*(X) = \{*\text{-det. slopes on } \partial X\}$, we get:

$$\boxed{\begin{array}{l} \underline{*\text{- Gluing Thm.}}\quad \text{Let } M = X_1 \cup_T X_2 \\ (X_i \text{ as above, } T = \partial X_1 = \partial X_2). \text{ Then} \\ \mathcal{D}_*(X_1) \cap \mathcal{D}_*(X_2) \neq \emptyset \Rightarrow M \text{ is } * . \end{array}}$$

For $* = NLS$: Hanselman-Rasmussen-Watson.

$* = LO$ Boyer-Clay.

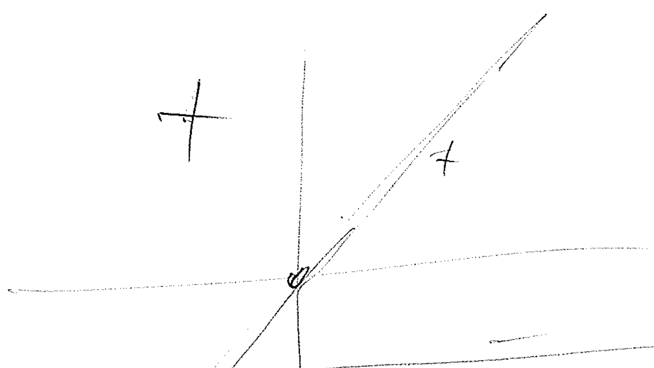
NLS: $\mathcal{D}_{NLS}(X) = \overline{\mathcal{S}(X) \setminus \mathcal{L}(X)}$

$\{$ slopes $\alpha \subset \partial X$: $X(\alpha)$ is an L-space $\}$.

LO: $<$ LO on G, $g \in G \rightsquigarrow$ LO $<_g$ on G:

$$a <_g b \iff ag < bg$$

$<$ LO on $\mathbb{Z} \times \mathbb{Z}$ determines a slope $s(<)$



$\alpha \in \mathcal{D}_{LO}(X) \iff \exists$ LO $<$ on $\pi_1(X)$ s.t.

$\forall g \in \pi_1(X)$: $\quad s(<_g | \pi_1(\partial X)) = \alpha$

(Boyer–Clay).

CTF : $\alpha \in \mathcal{D}_{CTF}(X) \iff \exists$ c.t. fol^n $\mathcal{F}$ on X s.t. $\mathcal{F} \cap \partial X$ has a leaf of slope α, & no Reeb annuli.

—— e.g. ——

λ = unique slope on ∂X s.t. $[\lambda] = 0 \in H_1(X; \mathbb{Q})$.

$\lambda \in \mathcal{D}_{\mathcal{F}}(X)$.

Proposition. X as above, suppose $[\lambda] = 0 \in H_1(X;\mathbb{Z})$; & let μ be a slope on ∂X s.t. $\underline{\mu \cdot \lambda = 1}$.

(1) $\mu \in \mathcal{D}_{NLS}(X)$.

(2) If X is a $\mathbb{Z}H(S^1 \times D^2)$, $\mu \in \mathcal{D}_{LO}(X)$.

(3) If X is fibered, $\mu \in \mathcal{D}_{CTF}(X)$.

This $\Rightarrow$ Thms 1, 2, 3.

E.g. Thm 1: $\underline{M} = X_1 \cup_T X_2$, a $\mathbb{Z}HS$,

Then X_1, X_2 are $\mathbb{Z}H(S^1 \times D^2)$s.

$H_1(M) = 0 \Rightarrow \lambda_1 \cdot \lambda_2 = 1$.

$\lambda_1 \in \mathcal{D}_{LO}(X_1)$,

$\lambda_1 \in \mathcal{D}_{LO}(X_2)$ (Prop (2)).

$\therefore$ M is LO by LO-Gluing Thm.

& M is CTF if X_2 is fibered, by CTF-Gluing Thm.

— □ —

$K = P(K_0)$; w = winding # of P in V; $d = (n, w)$; $m = n/d$.

$$\Sigma_n(K) = \Sigma_n(V, P) \cup \coprod_{i=1}^{d} X_m^{(i)}$$

$X_m^{(i)}$ copy of X_m = m-fold cyclic cover of $X = S^3 \setminus \mathring{N}(K_0)$.

$\mu \subset \partial X$ = meridian of K_0 $\rightsquigarrow$ $\mu_m \subset \partial X_m$

Propn. $\rightsquigarrow$ $\mu_m \in \mathcal{D}_*(X_m)$ for $* = NLS$ & LO,

& for CTF if K_0 is fibered.

Then an extension of $*$-Gluing Thm. to multislopes $\rightsquigarrow$ $\Sigma_n(K)$ is $*$.

(2) of Prop^n. $\lambda = \partial F$, $F \subset X$ compact or'tble. sfce. of min. genus.

Gabai: $\exists$ c.t. fol^n $\mathcal{F}_X$ on X with F as leaf.

$\rightsquigarrow$ c.t.f. $\mathcal{F}$ on $M = X \cup_\partial -X$;

$S = F \cup_\partial -F$ a leaf.

universal circle construction gives

$$\begin{tikzcd}
\pi_1(X) \arrow[r, phantom, "\overset{i}{\subset}"] \arrow[dr, dashed, "\tilde{\rho}"'] & \pi_1(M) \arrow[r, "\rho"] & \mathrm{Homeo}_+(S^1) \\
& \widetilde{\mathrm{Homeo}_+(S^1)} \arrow[ur] \arrow[r, phantom, "\subset"] & \mathrm{Homeo}_+(\mathbb{R})
\end{tikzcd}$$

where $\rho i(\pi_1(\partial X))$ has a fixed pt.

$\rho | \pi_1(S)$ semi-conj. to discrete faithful rep. $\to$ $PSL(2,\mathbb{R})$ $\Longrightarrow$ (Goldman) $\tilde{\rho}(\lambda)$ has no fixed pt.

$\therefore$ can choose $\tilde{\rho}$ so that $\tilde{\rho}(\mu)$ has a fixed pt.

$\Longrightarrow \mu \in \mathcal{D}_{LO}(X)$.